PRIME Problems

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Hippos

Martha is the chief hippopotamus caretaker at the Wild Animal Park in San Diego, California. She has just arrived at the cargo dock in the downtown harbor to pick up four members of the endangered species *hippopotamus mathematicus* recently rescued from African poachers. To complete the paperwork she needs to weigh them, but the only scale big enough to weigh a hippo is the truck scale that starts at 300 kilograms—more than any of the hippos weigh!

Martha is puzzled for a few minutes, but then gets the idea of weighing them in pairs. She thinks that if she gets the weight of every possible pair, she can figure out later the weights of the individual hippos. She measures the weights pair by pair, getting 312, 356, 378, 444, and 466 kilograms. As she tries to weigh the heaviest pair of hippos, the scale breaks.

(a) What was the weight of the last pair of hippos that broke the scale?

(b) What are the weights of all the individual hippos? Are you sure?

Notes about this problem: This problem can be solved algebraically as a system of equations. It also can be solved by trial and error. I have heard participants describe an interesting variety of strategies that were used to find solutions. There is not enough information to find a unique solution for the weights of individual hippos, but there is a unique solution for the weight of the heaviest pair that broke the scale. There are two possible systems of equations, depending on assumptions about which hippos (in order from lightest to heaviest) are paired for the middle combined weights. That can lead to a useful conversation about whether a system of linear equations can have exactly two unique solutions! (No, but there are two solutions to this problem because there are two possible assumptions that each lead to a different system of equations)

Method 1	Method 2
a + b = 312	a + b = 312
a + c = 356	a + c = 356
a + d = 378	a + d = 444
b + c = 444	b + c = 378
b + d = 466	b + d = 466
c + d = ?	c + d = ?

For both methods, c + d = 510

Method 1: a = 112, b = 200, c =244, d = 266

Method 2: a = 145, b = 167, c = 211, d = 299

Lockers

Two janitors at Central Middle School entertained themselves with the 250 lockers along the main corridor the evening after the last day of school. The first janitor opened every locker. The next janitor closed every second locker. The third time, the janitor changed the state (open or closed) of every third locker.

This process continued, with the state of every nth locker being changed during the nth pass, until the janitors had passed down the corridor 250 times. In the end, which lockers were open?

Notes about this problem: A good way to have students begin thinking about this problem is to have the class stand in a line. Tell students that when they are standing, that corresponds to a locker being open and when they are seated, the locker is closed. You can then model the problem up to the number of students in the class and they can take note of which students remain standing at the end. Don't have any discussion or hints after modeling—the point of the exercise is to see what solution they can find, and typically at least one group will at least notice which locker remain open (those numbered by a perfect square—1, 4, 9, 16, 25 at least in a typical class). Not as many will realize the reason: there are an even number of pairs of factors of non-perfect square numbers and an odd number of pairs of factors of perfect squares: for example, 15 is 1x15 and 3x5 while 16 is 1x16, 2x8, and 4x4.

Sums

Write a number in the circle and a number in the first square. For each of the remaining squares, add the number in the circle to the number in the preceding square to find the number in the next square.

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$$\square + \square + \square + \square + \square = 50$$

Notes about this problem: For example, 1 in the circle and 8 in the first square is one solution. Most people start solving this problem by trial and error. HS and college algebra students later will use equations and find the infinite number of solutions using that system. This is an arithmetic sequence whose common difference is in the circle and 5 terms are in the squares. A solution can be found with any number (including imaginary, irrational, transcendental) placed in either the circle or any one square EXCEPT the third square, which must be 10. If a similar problem with an even number of squares was posed solutions could be found with a number placed in any single position.